



On Boundary-Fixing Homotopies of Radial Homeomorphisms Defined on the Closed Planar Discs

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Abstract

In this work, homotopies of boundary-preserving homeomorphisms defined on the planar unit disc are studied. A set of radial homeomorphisms established in polar coordinates is considered. This set is subjected to rotating the circles, which have the same center, by an angle that depends in a continuous manner on the radius. However, these maps are homotopic to the identity map, the corresponding homotopies do not inevitably fix the boundary circle. For all parameter values, a point wise-fixed homotopy is defined between the homeomorphisms and the identity map. This establishes a concrete example and develops a technique for introducing homotopies restricted to a subspace. The outcomes support exploration of boundary-preserving transformations and planar homotopies in foundations of topology.

Keywords: Homotopy; Relative homotopy; Radial deformation; Boundary preservation

1. Introduction

In algebraic topology, the homotopy theory concerns subjects of continuous deformations of topological spaces. A homotopy can be defined as a set of continuous maps that continuously deforms a continuous map to another continuous map. This principle introduces a flexible approach for characterizing topological invariants of spaces and play an essential role to classify both of spaces and mappings through a deformation [1,2,3].

An important enhancement of this principal is a homotopy restricted to a subspace. Let $S \subseteq X$ be a subspace, a homotopy map is called relative to S when all its points classified as fixed points. The relative homotopy is especially crucial in planar coordinates and geometry fields, where boundary conditions are respected. These constructions arise naturally in planar homeomorphisms and in differential equations it can be studied in subject of boundary value problems [2,3].

Each of the closed unit disc and its boundary circle shows a traditional and well-developed setting for explaining homotopy-theoretic subject. Nontrivial homeomorphisms defined on that disc can be offered through polar maps which rotate the circles of radius r by an angle relying on that r . Although, these maps are homotopic to identity map, however the resulting homotopies do not conserve the boundary circle [4,5].

This observation inspires a problem this paper addressed: on planar discs with homotopies preserving boundary points. More rigorously, this work investigates a set of homeomorphisms defined on the closed disc influence as radial rotations and explore conditions of these maps will be homotopic to identity when we restrict that domain on the boundary circle [5,6,7].

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The essential objective of the paper is to construct a homotopy that fixes the boundary points by this deformation. The findings displayed here show a concrete example of boundary fixing homotopy and focus on technique for dealing with homotopies relative to subspaces [6,7].

The homotopy type of spaces of homeomorphisms on discs with punctures has also been investigated providing additional context for such radial maps [8, 9].

Expanding contexts, containing planar deformation problems in algebraic topology [6,7], and may relate to recent developments in hierarchically hyperbolic boundaries [8] and higher homotopy associativity [10,11,12].

Recent studies on boundary-preserving maps [13, 14] have further emphasized the importance of such relative deformations.

The connection with mapping class groups of the disc and with damped isotopies motivates a deeper examination of boundary-fixing homotopies [13, 14, 15].

2. Preliminaries

This section is to recall elementary notions and introductory concepts from both topology and homotopy which will serve results this paper.

2.1. Definition [2]

The closed unit disc D is the set of all points in the coordinate plane, that sum of squares is equal or less than one and its boundary circle C is just points that have radius equal to 1.

2.2. Definition [4]

A function h from a topological space X to a topological space Y is called continuous when $h^{-1}(A) \subseteq X$ is an open set in X for every open set A in Y .

2.3. Definition [3]

A continuous bijective function $h: X \rightarrow Y$ is called a homeomorphism if $h^{-1}: X \rightarrow Y$ is continuous.

2.4. Theorem [7]

If h is a bijective continuous function from a compact topological space X to a Hausdorff topological space Y , then h is a homeomorphism.

2.5. Definition [5]

Two continuous functions f and g from a topological space X to a topological space Y are called homotopic if there is a continuous function $H: X \times I \rightarrow Y$ satisfying the following statement: $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, for all $x \in X$, $I = [0, 1]$.

2.6. Definition [6]

Let S be subspace of a topological space X . A homotopy $H: X \times I \rightarrow Y$ is called relative to S when $H(s, t) = f(s)$, for all $s \in S$, $t \in I$.

2.7. Remark

Throughout this work, relative homotopy refers to C in definition 2.1 and D will be taken as closed unit disc.

2.8. Lemma [1]

Two maps f_1 and f_2 two continuous maps from a topological space X to spheres S^n , and each of them has never antipodal. Then there is a homotopy $H: X \times I \rightarrow S^n$ between f_1 and f_2 such that:

$$H(x, t) = \frac{tf_2(x) + (1-t)f_1(x)}{\|tf_2(x) + (1-t)f_1(x)\|}$$

3. Results and Discussion

In this section, we introduce the crucial results. Firstly, a homotopy between the identity map and a radial homotopy is constructed. Next we state this homotopy is taken such that it fixes points of C .

3.1. Theorem

Let $f: D \rightarrow D$ be a homeomorphism defined through

$f(r, \theta) = (r, \theta + 2\pi r)$ and $f(0) = 0$. Then there is a homotopy $H: D \times I \rightarrow D$, between map f and the identity map and for each $t \in I$, $H|_{D \times \{t\}}$ is homeomorphism.

Proof:

Define $H: D \times I \rightarrow D$ as follows, $H((r, \theta), t) = (r, \theta + 2\pi r(1 - t))$, $0 \leq t \leq 1$.

- Continuity of H results from it is a polynomial for variables θ , r and t .
- Homotopy of H comes from, if $t = 0$, then $H((r, \theta), 0) = (r, \theta + 2\pi r) = f(r, \theta)$, and if $t = 1$, then $H((r, \theta), 1) = (r, \theta + 0) = (r, \theta)$.
- Now, $D \times I$ and D are compact and Hausdorff spaces, respectively in $\mathbb{R}^2$, thus by theorem 2.4, $H|_{D \times \{t\}}$ is homeomorphism for all t [7].

3.2. Theorem

If f is the map in theorem 3.1 above, then it is homotopic to the identity map relative to C .

Proof:

The map f in the theorem 3.1 rotates the boundary points on C (where $r = 1$) by 2π . This means that f behaves as identity on C . But the obtained homotopy, for all t does not fix C , so we must find a different homotopy map.

Let us define $\tilde{H}: D \times I \rightarrow D$ as above. Since $0 \leq t \leq 1$, this means that $\tilde{H}$ is continuous map in the variables t , r and θ . Therefore $\tilde{H}((r, \theta), 0) = f(r, \theta)$ and $\tilde{H}((r, \theta), 1) = (r, \theta)$. This means that $\tilde{H}((r, \theta), 1)$ represents the identity map for the variables r, θ . Lastly, we see that $\tilde{H}((1, \theta), t) = (1, \theta)$, indicating it fixes C and for all values t .

Thus we have produced a boundary-fixing homotopy, which is a concrete example of the general theory developed in [13, 14]. Such homotopies are essential for understanding the mapping class group of the disc [13] and for applications to fixed point problems [15].

3.3. Theorem

If g is a continuous map defined on unit circle C and not homotopic to identity map, then for some $x \in C$, $g(x) = -x$.

Proof:

Suppose that C is the unit circle and $g(x) \neq x$. Thus we can apply lemma 2.8 to h and $g(x) = x$. Under this hypothesis results $h(x)$ and $g(x) = x$ never antipodal. This means $h(x)$ and $g(x)$ are homotopic and also g is not homotopic to identity. So there is at least one point x on the circle C such that $g(x) = -x$. This result is related to classical fixed point theorems; for further discussions on fixed point properties on discs and circles we refer to [12, 15].

4. Conclusion

This examined homotopies of radial homeomorphisms, where D is their domain, considering homotopies that preserve points of the circle C . Basic radial homeomorphisms are constructed and explained that despite these maps homotopic to identity map, the resulting homotopies do not surely fix the points of C .

To tackle this problem, a homotopy is defined to refine the radial deformation by a method that depends on the radius. This method entails homotopy relative to C .

In addition, a condition of presence of fixed points on C is formulated, showing importance of antipodal reaction in the relative homotopy. These results support understanding deformations in planar fields and suggest a manner for forming relative homotopy. Planar homeomorphisms can be extended relying on these methods to be a basis in relative homotopy. Recent advances in boundary-fixing deformations [13, 14], fixed point theory [12, 15], and the homotopy type of homeomorphism spaces [9] provide a broader context for our work. Moreover, connections with equivariant homotopy [11] and higher homotopy associativity [10] may inspire further investigations.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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